VIETNAM NATIONAL UNIVERSITY HO CHI MINH CITY
UNIVERSITY OF SCIENCE

DO HOANG VIET

INTERSECTION GRAPHS OF GENERAL LINEAR GROUPS

SENIOR THESIS

Ho Chi Minh City - 2022

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A SENIOR THESIS

ON ALGEBRA AND NUMBER THEORY

ADVISOR
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Ho Chi Minh City - 2022

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Introduction

Graph theory is a mathematical field in which researchers study the connection between objects and submit applications widely in scientific areas. In terms of the relation between graph theory and algebra, intersection graphs of groups were introduced by Csakany and Pollak in 1969 [CP69] and the topic has been developed until now. One of many milestones in the progress was the results of Rulin Shen [She10], he showed the diameters of the intersection graphs of several finite non-simple groups are at most four. This inspired us to cope with the similar problem for general linear groups for the purpose of finding exactly values of the diameters of their intersection graphs.

Besides, since most of previous results had been properties of finite groups, we raise naturally a question of the structure of intersection graphs of infinite ones. Indeed, with the aim of broadening perspective on infinite general linear groups, we researched the end-equivalence relation on the set of rays of infinite graphs which are notions created by the German graph theorist Rudolf Halin [Hal64].

In the dissertation, a more detailed version of our paper [BV21], we show how to handle the above problems. Regarding the first question, we proved that the precise value of diameter of the intersection graph of general linear groups is either two or three. In terms of the infinite groups, we showed that the intersection graph of any general linear group over an infinite field is one-end.

Here is a more detailed account of this dissertation.

Chapter 1 introduces basic terminologies of graph and group theory. Specially, the notions and properties of rays and Noetherian groups play the most important role.

In chapter 2, we answer two main questions raise above by giving completed proofs.

Chapter 1

Background

1.1 Some graph theory

1.1.1 Basic definitions

In the dissertation, we just pay attention to *simple undirected graphs* defined as follows.

Definition 1.1.1. A *(simple undirected) graph* Γ is an ordered pair $(\mathcal{V}, \mathcal{E})$ comprising:

(i) A non-empty set $\mathcal{V}$ called the set of *vertices*;

(ii) A set $\mathcal{E} \subseteq \{\{u, v\} \mid u, v \in \mathcal{V}, u \neq v\}$ called the set of *edges*.

If $\{u, v\}$ is an edge of Γ, which means that $\{u, v\} \in \mathcal{E}$, then we call that u and v are *adjacent* and often illustrate the relation between them as follows:

$$u \text{ ——— } v.$$

We call Γ *finite* if it has finitely many vertices. Conversely, if Γ has infinitely many vertices and edges, then it is called an *infinite* graph.

Definition 1.1.2. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph and let $u, v \in \mathcal{V}$ be two vertices. A *path* from u to v is defined to be a sequence of finitely many vertices in which the first and the last vertices are u and v, respectively, and two consecutive vertices are adjacent as follows:

$$u = v_0 \text{ ——— } v_1 \text{ ——— } \cdots \text{ ——— } v_{n-1} \text{ ——— } v_n = v.$$

The number n is called the *length* of the path. A path is called *simple* if all of its vertices are distinct. If we do not pay attention to intermediate vertices, then we can write

$$u \rightsquigarrow v.$$

Notation 1.1.3. We do not avoid cases in which the length of a path is zero by permitting a path of length zero is a vertex. Conversely, any vertex can be seen as a zero-length path.

Definition 1.1.4. Let Γ be a graph. For two paths Γ_1 and Γ_2 of Γ as follows:

$$\Gamma_1 : \qquad v_0 \text{ --- } v_1 \text{ --- } \cdots \text{ --- } v_{n-1} \text{ --- } v_n,$$

$$\Gamma_2 : \qquad v_n = u_0 \text{ --- } u_1 \text{ --- } \cdots \text{ --- } u_{k-1} \text{ --- } u_k.$$

We define the *concatenation* of Γ_1 and Γ_2 to be the path

$$\Gamma_1 \circ \Gamma_2 : \qquad v_0 \text{ --- } v_1 \text{ --- } \cdots \text{ --- } v_n \text{ --- } u_1 \text{ --- } \cdots \text{ --- } u_k.$$

Definition 1.1.5. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph and let u, v be two vertices. If there are at least one path from u to v, then the *distance* from u to v, denoted by $d(u, v)$, is defined to be the length of the shortest one among such paths. Otherwise, we put $d(u, v) = \infty$. A graph $\Gamma = (\mathcal{V}, \mathcal{E})$ is called *connected* if $d(u, v) < \infty$ for all $u, v \in \mathcal{V}$.

Definition 1.1.6. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a connected graph. We define the *diameter* of Γ, denoted by $\delta(\Gamma)$, to be the supremum of the set of distances between any two vertices of Γ, that is

$$\delta(\Gamma) = \sup\{d(u, v) \mid u, v \in \mathcal{V}\} \leq \infty.$$

Definition 1.1.7. A graph $\Gamma = (\mathcal{V}, \mathcal{E})$ is called a *complete* graph if $\delta(\Gamma) = 1$, which means that $\mathcal{E} = \{\{u, v\} \mid u, v \in \mathcal{V}, u \neq v\}$.

1.1.2 Rays of an infinite graph and the end-equivalence relation

In this subsection, we focus on infinite connected graphs.

Definition 1.1.8. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be an infinite connected graph. A *ray* of Γ is an one-sided countably infinite chain of distinct vertices of $\mathcal{V}$ such that consecutive vertices are adjacent, that is

$$(v_i)_{i \geq 0} : \qquad v_0 \text{ --- } v_1 \text{ --- } \cdots \text{ --- } v_n \text{ --- } \cdots.$$

Definition 1.1.9. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be an infinite connected graph. Two rays $(v_i)_{i \geq 0}$ and $(u_i)_{i \geq 0}$ of Γ are called *disjoint* if they have no vertex in common, which means that the set $\{v_i \mid i \geq 0\} \cap \{u_i \mid i \geq 0\}$ is the empty set.

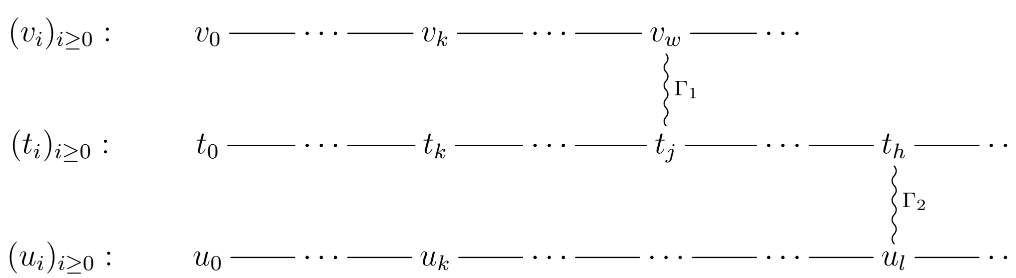
Definition 1.1.10. Let Γ be an infinite connected graph and let A and B be two sets of vertices. A **finite** set of vertices F is said to *separate* A and B, or equivalently A is *separated* from B by F, if any path from an arbitrary vertex in A to some vertex in B always contains a vertex in F.

Definition and Proposition 1.1.11. Let Γ be an infinite connected graph. Two rays $(v_i)_{i\geq 0}$ and $(u_i)_{i\geq 0}$ of Γ are called *end-equivalent*, denoted by $(v_i)_{i\geq 0} \equiv (u_i)_{i\geq 0}$, if the sets $\{v_i \mid i \geq 0\}$ and $\{u_i \mid i \geq 0\}$ cannot be separated by any **finite** set of vertices of Γ. The end-equivalence of rays is an equivalence relation on the set of all rays of Γ.

Proof. Let $(v_i)_{i\geq 0}$ be an arbitrary ray of Γ. By Definition 1.1.10, any finite set that can separate $\{v_i \mid i \geq 0\}$ and itself must contain some vertex of each path of length zero from v_i to itself, for each $i \geq 0$. Therefore, the finite set must contain all of v_i's. However, the set $\{v_i \mid i \geq 0\}$ is infinite, so there is not any finite set of vertices that can separate the ray and itself. As a result, the reflexivity holds.

The symmetry is verified routinely. We claim that the transitivity of the relation also holds. Let $(v_i)_{i\geq 0}$, $(u_i)_{i\geq 0}$ and $(t_i)_{i\geq 0}$ be three distinct rays of Γ satisfying $(v_i)_{i\geq 0} \equiv (t_i)_{i\geq 0}$ and $(u_i)_{i\geq 0} \equiv (t_i)_{i\geq 0}$. For arbitrary finite set F of vertices of Γ, it is sufficient to show that there exists a path from v_i to u_j for some $i, j \geq 0$ such that this path does not contain any vertex of F. We prove the transitivity as follows.

Fix a finite set F. Therefore, there exists $k \geq 0$ such that $F \cap \{v_i, t_i, u_i \mid i > k\} = \emptyset$. By hypothesis, the finite set $F \cup \{v_i, u_i, t_i \mid 0 \leq i \leq k\}$ cannot separate $\{v_i \mid i \geq 0\}$ and $\{t_i \mid i \geq 0\}$, which enables us to find a path Γ_1 from some v_w to some t_j such that Γ_1 does not contain any vertex of $F \cup \{v_i, t_i, u_i \mid 0 \leq i \leq k\}$. As a result, it must satisfy $w > k$ and $j > k$. Similarly, by considering the finite set $F \cup \{v_i, t_i, u_i \mid 0 \leq i \leq j\}$, we also obtain a path Γ_2 from t_h to u_l for some $h, l > j$ such that Γ_2 does not contain any vertex of $F \cup \{v_i, t_i, u_i \mid 0 \leq i \leq j\}$.



We consider the path $\Gamma_1 \circ (t_z)_{j \le z \le h} \circ \Gamma_2$ from v_w to u_l. Because $F \cap \{v_i, u_i, t_i \mid i > k\} = \emptyset$, we have $\Gamma_1 \circ (t_z)_{j \le z \le h} \circ \Gamma_2$ does not contain any vertex of F. To sum up, we obtain $(v_i)_{i \ge 0} \equiv (u_i)_{i \ge 0}$. The proof is completed. □

Notation 1.1.12. In Definition 1.1.10, if A is a finite set of vertices, we have A is separated from A by itself! This means that the reflexivity is violated if the end-equivalence relation is established between finite sets of vertices. In the other words, the end-equivalence is a characteristic of infinite graphs.

Definition 1.1.13. Let Γ be an infinite connected graph. An end-equivalence class of rays of Γ is called an *end* of Γ. We call Γ *one-end* if Γ has exactly one end. If an end contains infinitely many disjoint rays, then we say the end to be *thick*.

We have an equivalent definition of end-equivalence relation in the next proposition. It is more simple to understand and helpful to use, but we find it difficult to show its transitivity. Therefore, we use the notion in Definition 1.1.11 to establish the relation and utilize the following condition to cope with the problem in the final section of the dissertation.

Proposition 1.1.14. Let Γ be an infinite connected graph and let $(v_i)_{i \ge 0}$ and $(u_i)_{i \ge 0}$ be two rays of Γ. Then two following conditions are equivalent:

(i) $(v_i)_{i \ge 0} \equiv (u_i)_{i \ge 0}$;

(ii) there exists a ray $(t_i)_{i \ge 0}$ such that $\{v_i \mid i \ge 0\} \cap \{t_i \mid i \ge 0\}$ are infinite as well as $\{u_i \mid i \ge 0\} \cap \{t_i \mid i \ge 0\}$.

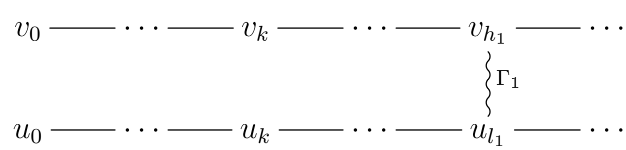
Proof. (i) $\Rightarrow$ (ii): If the set $\{v_i \mid i \ge 0\} \cap \{u_i \mid i \ge 0\}$ is infinite, then we can choose $t_i = v_i$ for every $i \ge 0$ to complete the proof for this case.

Otherwise, there is a number k such that $\{v_i \mid i > k\} \cap \{u_i \mid i > k\} = \emptyset$. Put $F_1 = \{v_i, u_i \mid 0 \le i \le k\}$. By hypothesis, the fact that the finite set F_1 cannot separate $\{v_i \mid i \ge 0\}$ and $\{u_i \mid i \ge 0\}$ enables us to have at least one path from some vertex of $(v_i)_{i \ge 0}$ to some of $(u_i)_{i \ge 0}$ such that each of them does not contain any vertex of F_1. Among such paths, we choose the shortest path Γ_1 from v_{h_1} to u_{l_1}. We obtain three basic properties of the path Γ_1 as follows:

(1) It must satisfy that $h_1 > k$ and $l_1 > k$ because Γ_1 does not contain any vertex of the set $F_1 = \{v_i, u_i \mid 0 \leq i \leq k\}$;

(2) Γ_1 is simple because we can replace Γ_1 by a shorter path if it has two repeated vertices, which in turn contradicts the shortest length of Γ_1.

(3) Except the first vertex v_{h_1} and the last vertex u_{l_1}, Γ_1 does not contain any vertex of two rays $(v_i)_{i\geq 0}$ and $(u_i)_{i\geq 0}$. In fact, if there exists an intermediate vertex belonging to either $(v_i)_{i\geq 0}$ or $(u_i)_{i\geq 0}$, then we can replace Γ_1 by a shorter path. This in turn contradicts the shortest length of Γ_1. Therefore, $\Gamma_1 \cap \{v_i, u_i \mid i \geq 0\} = \{v_{h_1}, v_{l_1}\}$.



Inductively, for each $z \geq 1$, we assume that the path Γ_z from v_{h_z} to u_{l_z} was constructed, we put

$$F_{z+1} = \{v_i, u_i \mid 0 \leq i \leq \max\{h_z, l_z\}\} \cup \bigcup_{1\leq i\leq z} \Gamma_i.$$

It is routine to verify that F_{z+1} is finite, which enables us to choose a path Γ_{z+1} from $v_{h_{z+1}}$ to $u_{l_{z+1}}$ in the same way of choosing Γ_1. Repetition of the above arguments helps us to show three following conditions:

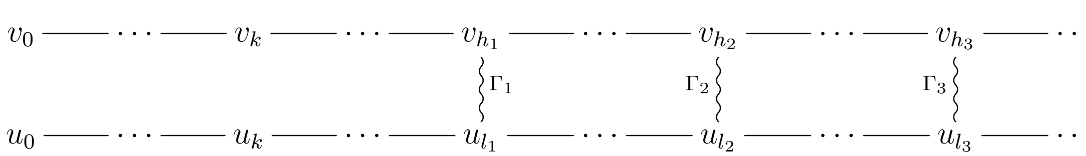
(1') $h_{z+1} > \max\{h_z, l_z\} \geq h_z$ and $l_{z+1} > \max\{h_z, l_z\} \geq l_z$, for every $z \geq 1$;

(2') Γ_z is simple for each $z \geq 1$;

(3') $\Gamma_z \cap \{v_i, u_i \mid i \geq 0\} = \{v_{h_z}, u_{l_z}\}$.

On the other hand, how we set F_{z+1} allows us to obtain an additional property:

(4') $\Gamma_{z+1} \cap (\bigcup_{1\leq i\leq z} \Gamma_i) = \emptyset$, for every $z \geq 1$.



Now, we consider the path

$$(t_i)_{i\geq 0} := \Gamma_1 \circ (u_i)_{l_1\leq i\leq l_2} \circ \Gamma_2 \circ (v_i)_{h_2\leq i\leq h_3} \circ \Gamma_3 \circ \cdots .$$

We claim that $(t_i)_{i\geq 0}$ is a ray, which means that vertices of $(t_i)_{i\geq 0}$ are distinct. In fact, for two arbitrary vertices t_j and $t_{j'}$, there are totally five following general cases.

- Both t_j and $t_{j'}$ belong to $(v_i)_{h_2\leq i\leq h_3}\cup(v_i)_{h_4\leq i\leq h_5}\cup\cdots\subseteq\{v_i \mid i>k\}$. Because $(v_i)_{i\geq 0}$ is a ray, t_j must be different from $t_{j'}$;

- Both t_j and $t_{j'}$ belong to $(u_i)_{l_1\leq i\leq l_2}\cup(u_i)_{l_3\leq i\leq l_4}\cup\cdots\subseteq\{u_i \mid i>k\}$. Because $(u_i)_{i\geq 0}$ is a ray, t_j must be different from $t_{j'}$;

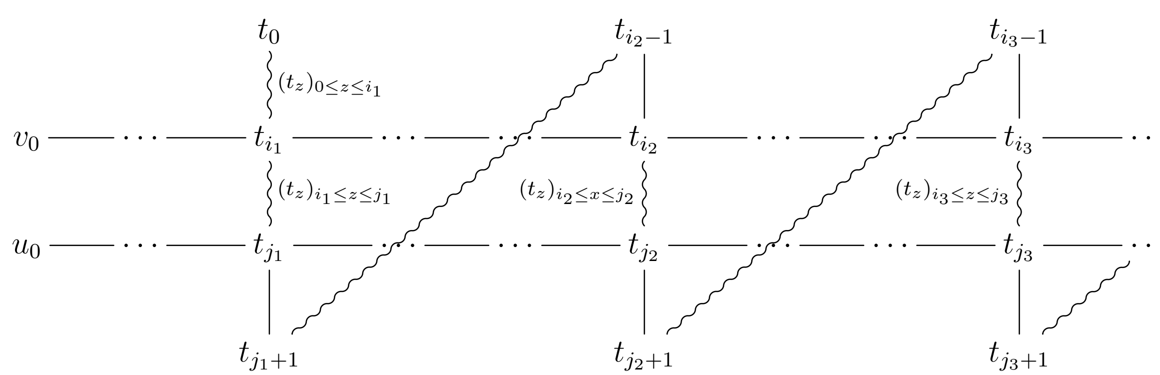
- If
$$t_j\in(v_i)_{h_2\leq i\leq h_3}\cup(v_i)_{h_4\leq i\leq h_5}\cup\cdots\subseteq\{v_i \mid i>k\}$$
and
$$t_{j'}\in(u_i)_{l_1\leq i\leq l_2}\cup(u_i)_{l_3\leq i\leq l_4}\cup\cdots\subseteq\{u_i \mid i>k\},$$
then it must satisfy $t_j\neq t_{j'}$ because $\{v_i \mid i>k\}\cap\{u_i \mid i>k\}=\emptyset$;

- When $t_j\in\Gamma_p\setminus\{v_i,u_i \mid i>k\}$ and $t_{j'}\in\Gamma_{p'}\setminus\{v_i,u_i \mid i>k\}$, there are two following particular cases. If $p=p'$, then we obtain $t_j\neq t_{j'}$ according to (2'). Otherwise, when $p\neq p'$, we also reach the same conclusion by utilizing (4');

- If $t_j\in\{v_i,u_i \mid i>k\}$ and $t_{j'}\in\Gamma_{p'}\setminus\{v_i,u_i \mid i>k\}$, then we obtain $t_j\neq t_{j'}$ obviously.

As a consequence, $(t_i)_{i\geq 0}$ is a ray. Finally, it is routine to verify that $\{v_i \mid i\geq 0\}\cap\{t_i \mid i\geq 0\}$ and $\{u_i \mid i\geq 0\}\cap\{t_i \mid i\geq 0\}$ are infinite sets.

(ii) $\Rightarrow$ (i): For each $k\geq 1$, the hypothesis enables us to choose t_{i_k} and t_{j_k} such that $t_{i_k}\in(v_i)_{i\geq 0}$ and $t_{j_k}\in(u_i)_{i\geq 0}$ and $i_k<j_k<i_{k+1}$.



We have the paths $(t_z)_{i_k \leq z \leq j_k}$'s are disjoint because $(t_i)_{i\geq 0}$ is a ray. If F is a finite set that can separate $(v_i)_{i\geq 0}$ and $(u_i)_{i\geq 0}$, then F must contain at least one vertex of each path $(t_z)_{i_k \leq z \leq j_k}$ for each $k \geq 1$. As a consequence, F must be infinite, a contradiction! To sum up, it must satisfy that $(v_i)_{i\geq 0} \equiv (u_i)_{i\geq 0}$. The proof is completed. □

Remark 1.1.15. By Proposition 1.1.14, we can realize that a ray $(v_i)_{i\geq 0}$ is usually end-equivalent to its "tail" $(v_i)_{i\geq k}$, for all $k \geq 0$.

1.2 Group theory

1.2.1 Basic notions

Definition 1.2.1. A *group* G is a non-empty set, together with a binary operation $\circ : G \times G \to G$, namely a *multiplication*, such that the following axioms hold:

(i) The *associative law*: for every $x, y, z \in G$,

$$(x \circ y) \circ z = x \circ (y \circ z).$$

(ii) There is an unique element $e \in G$, called the *identity*, such that

$$e \circ x = x \circ e = x,$$

for every $x \in G$.

(iii) Each $x \in G$ has an unique *inverse* which is an element $y \in G$ satisfying

$$x \circ y = y \circ x = e.$$

We shall follow a custom of suppressing the symbol "$\circ$" and writing either xy or $x \cdot y$ in place of $x \circ y$ and call it the *product* of x and y. Furthermore, we denote the identity and the inverse of each element x of a group G by 1 (or 1_G) and x^{-1}, respectively.

If the group G consists of finitely many elements, the number of its elements is called the *order* of the group, denoted by $|G|$. Otherwise, we say that G is infinite.

Two elements x and y of a group G are said to *commute* if $xy = yx$. The *center* of a group G, denoted by $Z(G)$, is defined to be the set consisting of all elements commuting with every elements of G, which means that

$$Z(G) = \{z \in G \mid zx = xz \text{ for every } x \in G\}.$$

A group G is called *abelian* if $Z(G) = G$.

Let G be a group and let $x \in G$. If there exists the smallest positive integer k satisfying $x^k = 1$, then k is called the *order* of x. Otherwise, we say that x has infinite order.

Example 1.2.2. The *general linear group* of degree $n \geq 2$ over a field $\mathbb{F}$, denoted by $\mathrm{GL}_n(\mathbb{F})$, is the set of $n \times n$ invertible matrices over the field $\mathbb{F}$, together with the operation of ordinary matrix multiplication. This forms a group with identity matrix as the identity of the group, and the inverse of an element (being a matrix) is its ordinary inverse.

According to linear algebra, $\mathrm{GL}_n(\mathbb{F})$ consists of all matrices with non-zero determinant and it is non-abelian. The center of $\mathrm{GL}_n(\mathbb{F})$ is the set of all scalar matrices of non-zero determinant, which means that

$$Z(\mathrm{GL}_n(\mathbb{F})) = \{\alpha \mathrm{I}_n \mid \alpha \in \mathbb{F}^*\}.$$

If $\mathbb{F}$ is infinite, then $\mathrm{GL}_n(\mathbb{F})$ is obviously infinite for every $n \geq 2$.

Otherwise, if $|\mathbb{F}| = q$, then the order of $\mathrm{GL}_n(\mathbb{F})$ is

$$\prod_{k=0}^{n-1}(q^n - q^k) = (q^n - 1)(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1}).$$

Definition 1.2.3. Let G be a group. A subset H of G is a *subgroup* if

(i) $1 \in H$;

(ii) whenever $x, y \in H$, then $xy \in H$;

(iii) if $x \in H$, then $x^{-1} \in H$.

If H is a subgroup of G, then we write $H \leq G$. It is obvious that $\{1\} \leq G$, it is called the *trivial* subgroup of G. On the other hand, G is also a subgroup of G itself. If the subgroup H is different from G, then H is called a *proper* subgroup of G, denoted by $H < G$.

Example 1.2.4. The *special linear group* of degree n over a field $\mathbb{F}$, denoted by $\mathrm{SL}_n(\mathbb{F})$, is the set of $n \times n$ matrices with determinant 1, with the group operations of ordinary multiplication and matrix inversion. This forms a subgroup of the general linear group of the same degree over the field $\mathbb{F}$, i.e. $\mathrm{SL}_n(\mathbb{F}) \leq \mathrm{GL}_n(\mathbb{F})$. When $\mathbb{F}$ contains at least three elements, we have $\mathrm{SL}_n(\mathbb{F}) < \mathrm{GL}_n(\mathbb{F})$.

If $\mathbb{F}$ is infinite and $n \geq 2$, then $\mathrm{SL}_n(\mathbb{F})$ is infinite. Otherwise, if $|\mathbb{F}| = q$ and $n \geq 2$, then the order of $\mathrm{SL}_n(\mathbb{F})$ is

$$\frac{1}{q-1}\prod_{k=0}^{n-1}(q^n - q^k) = \frac{1}{q-1}(q^n-1)(q^n-q)(q^n-q^2)\dots(q^n-q^{n-1}).$$

Proposition 1.2.5. (i) The intersection $\bigcap_{i \in I} H_i$ of any family of subgroups of a group G is again a subgroup of G. In particular, if $H, K \leq G$, then $H \cap K \leq G$.

(ii) If G_i is a subgroup of a group G for every $i \geq 1$ such that $G_i \leq G_j$ for all $i \leq j$, then their union $\bigcup_{i \geq 1} G_i$ is also a subgroup of G.

Proof. The proof is omitted on purpose. □

Definition 1.2.6. Let G be a group and let $X \subseteq G$. We define $\langle X \rangle$ to be the intersection of the family of all subgroups of G containing X.

We call that $\langle X \rangle$ is *generated by* X. If a subgroup of G is generated by a set of finitely many elements, then the subgroup is said to be *finitely generated*.

For two subsets X and Y of a group G, we define the *product* of X and Y to be the set $XY = \{xy \mid x \in X, y \in Y\}$. If H is a subgroup of G and $x \in G$, then xH and Hx are called a *left coset* and a *right coset* of H in G, respectively. We denote the set of left cosets of H in G by G/H.

Theorem 1.2.7 (Lagrange's Theorem)**.** *If H is a subgroup of a finite group G, then $|H|$ divides $|G|$.*

Proof. See [**Ro10**]. □

Definition 1.2.8. Let G be a group and let $H \leq G$ $(H < G)$. If $x^{-1}Hx \subseteq H$ for every $x \in G$, then H is a *normal* subgroup of G and we denote $H \trianglelefteq G$ $(H \triangleleft G)$.

It is routine to verify an operation between left cosets of H in G that $(xH)(yH) = (xyH)$ for every $x, y \in G$. This is the motivation for establishing the following definition.

Definition and Proposition 1.2.9. Let H be a normal subgroup of a group G. The set G/H equipped with the above operation is a group called the *quotient* group of G mod H. The identity of G/H is H and the inverse of a coset xH is $x^{-1}H$.

Proof. See [**Ro10**]. □

Definition 1.2.10. Let $(G, \circ)$ and $(G', \circ')$ be two groups. A mapping $f : G \to G'$ is called a *group homomorphism* (or *homomorphism*) if

$$f(x \circ y) = f(x) \circ' f(y), \text{ for every } x, y \in G.$$

If a group homomorphism is a bijection, then it is called an (group) *isomorphism*. Two groups G and G' are called *isomorphic* , denoted by $G \simeq G'$, if there is an isomorphism between them.

We define the *kernel* of f to be $\mathrm{Ker} f = \{x \in G \mid f(x) = 1_{G'}\}$, and the *image* of f to be $\mathrm{Im} f = \{f(x) \mid x \in G\}$.

Proposition 1.2.11. Let $f : G \to G'$ be a homomorphism. Then we have following results:

(i) $\mathrm{Im} f \leq G'$ and $\mathrm{Ker} f \trianglelefteq G$;

(ii) f is injective if and only if $\mathrm{Ker} f = \{1_G\}$;

(iii) $G/\mathrm{Ker} f \simeq \mathrm{Im} f$.

Proof. See [**Ro10**]. □

Definition 1.2.12. Let G be a group and let $x \in G$. Then the group $\langle x \rangle := \{x^n \mid n \in \mathbb{Z}\}$ is called a *cyclic subgroup* of G *generated* by x. A group G is called *cyclic* if $G = \langle x \rangle$ for some $x \in G$. In this case, x is called a *generator* of G.

Proposition 1.2.13. (i) Every group of prime order is cyclic.

(ii) Let G be a finite group and let $x \in G$. Then the order of $\langle x \rangle$ is the order of x and is a divisor of $|G|$.

(iii) If $G = \langle x \rangle$ is a cyclic group of order n, then $G = \langle x^k \rangle$ if and only if $(n, k) = 1$.

(iv) A cyclic group G of order n has an unique subgroup of order d for every divisor d of n.

Proof. See [**Ro10**]. $\square$

1.2.2 Noetherian groups

Definition 1.2.14. A group G is said to satisfy the *ascending chain condition (ACC)* if every ascending chain of subgroups

$$G_1 \leq G_2 \leq \cdots \leq G_n \leq \cdots$$

stops, which means that there is an integer k such that $G_k = G_{k+1} = G_{k+2} = \cdots$.

Definition and Proposition 1.2.15. For a group G, the following conditions are equivalent:

(i) G satisfies the ACC;

(ii) G satisfies the *maximum condition*: Every nonempty family $\mathcal{M}$ of subgroups of G has a maximal element, which means that there is some $H \in \mathcal{M}$ such that there is no $K \in \mathcal{M}$ with $H < K$;

(iii) Every subgroup of G is finitely generated.

If the group G satisfies these conditions, then G is called a *Noetherian* group.

Proof. (i) $\Rightarrow$ (ii): Let $\mathcal{M}$ be a nonempty family of subgroups of G and assume that $\mathcal{M}$ does not have any maximal element. We are going to construct inductively an ascending chain of subgroups in $\mathcal{M}$. Choose $H_1 \in \mathcal{M}$. For $i \geq 1$, since H_i is not a maximal element in $\mathcal{M}$, there exists $H_{i+1} \in \mathcal{M}$ such that $H_i < H_{i+1}$. As a result, we obtain an ascending chain

$$H_1 < H_2 < \cdots < H_n < \cdots$$

of subgroups of G. However, the chain does not stop, which contradicts the ACC.

(ii)$\Rightarrow$ (iii): For $H \leq G$, we define $\mathcal{M}$ to be the family of all finitely generated subgroups of H. Since $\langle 1 \rangle \leq H$, we have $\langle 1 \rangle \in \mathcal{M}$, so $\mathcal{M} \neq \emptyset$. By the hypothesis, there exists a maximal element K of $\mathcal{M}$. If $K < H$, then there is some element $h \in H \setminus K$, implying

$K < \langle K, h \rangle \leq H$. On the other hand, K is finitely generated, so is $\langle K, h \rangle$, which enables $\langle K, h \rangle$ to belong to $\mathcal{M}$. This contradicts the maximality of K. As a result, it must satisfy that $K = H$, so H is finitely generated.

(iii) $\Rightarrow$ (i): Let

$$H_1 \leq H_2 \leq \cdots \leq H_n \leq \cdots$$

be an ascending chain of subgroups of G. By (ii) of Proposition 1.2.5, the union set

$$H = \bigcup_{i \geq 1} H_i$$

is a subgroup of G. By the hypothesis, H is finitely generated by a set of elements $h_1, \ldots, h_q$. As a result, there are some subgroup H_{n_i} such that $h_i \in H_{n_i}$ for every $1 \leq i \leq q$. Choose N being greater than all n_i's. We obtain that $h_i \in H_N$ for all $1 \leq i \leq q$, which implies $H \leq H_N$. Therefore, $H = H_N$, or equivalently $H_N = H_{N+1} = \cdots$. The proof is completed. □

Chapter 2

Intersection graphs of general linear groups

At the beginning of this chapter, we recall two results in field theory.

Proposition 2.0.1. Let $\mathbb{F}$ be a finite field. Then $|\mathbb{F}| = p^m$ for some $m \geq 1$ where p is the characteristic of $\mathbb{F}$.

Proof. See [**Ro10**]. □

Proposition 2.0.2. If $\mathbb{F}$ is a field and G is a finite subgroup of the multiplicative group $\mathbb{F}^*$, then G is cyclic. In particular, if $\mathbb{F}$ is finite, then $\mathbb{F}^*$ is a cyclic group. In this case, a generator of $\mathbb{F}^*$ is called a *primitive element* of $\mathbb{F}$.

Proof. See [**Ro10**]. □

2.1 The completeness of intersection graphs of groups

First of all, let us pay attention to the definition of intersection graphs of groups, the main object of the dissertation.

Definition 2.1.1. Let G be a (not necessarily finite) group. The *intersection graph* of G, denoted by $\Gamma(G) = (\mathcal{V}_G, \mathcal{E}_G)$, is the graph defined as follows:

(i) The set of vertices $\mathcal{V}_G$ consists of all non-trivial proper subgroups of G;

(ii) For $A, B \in \mathcal{V}_G$, A, B are adjacent if and only if $A \neq B$ and $A \cap B \neq \{1\}$.

Remark 2.1.2. Only if the set of vertices $\mathcal{V}_G$ has more than one vertex is the intersection graph $\Gamma(G)$ established. This means that G has more than one non-trivial proper subgroup. In order to ensure the necessary condition, the order of G is not a prime number, nor is it the square of a prime number.

The relation between subgroups of prime order of a finite group G and the completeness of its intersection graph is illustrated in the following proposition.

Proposition 2.1.3. The intersection graph of a non-trivial finite group G is complete if and only if G has only one subgroup of prime order.

Proof. If G has only one subgroup H of prime order, then every non-trivial proper subgroup of G contains H. Therefore, the intersection of every two distinct non-trivial proper subgroups of G contains H. As a result, every two distinct vertices of $\Gamma(G)$ are adjacent.

Conversely, assume that $\Gamma(G)$ is complete. If there are two distinct non-trivial subgroups H_1 and H_2 of prime order, then $H_1 \cap H_2 = \{1\}$. Therefore, H_1 and H_2 are not adjacent, which contradicts the completeness of $\Gamma(G)$. As a result, G has only one subgroup of prime order. The proof is completed. □

From the later section, we can realize that the subgroups of prime order can be considered as the skeleton of big picture of the dissertation (except the last section about infinite intersection graphs).

Every group we consider for the rest of the dissertation has more than one group of prime order. By Proposition 2.1.3, its intersection graph is not complete, and we also have a result related to the diameter of the intersection graph in this case.

Proposition 2.1.4. For a finite group G, if $\Gamma(G)$ is connected and not complete, then

$$\delta(\Gamma(G)) = \max\{d(A, B) \mid A, B \in \mathcal{V}_G \text{ of prime order}\}.$$

Proof. Because G is finite, $\Gamma(G)$ is also finite, which enables us to obtain the equality

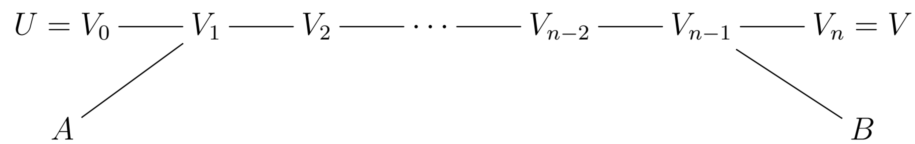
$$\delta(\Gamma(G)) = \sup\{d(U, V) \mid U, V \in \mathcal{V}_G\} = \max\{d(U, V) \mid U, V \in \mathcal{V}_G\} = n \geq 2.$$

This means that there are two vertices U and V of $\Gamma(G)$ whose distance equals the diameter of $\Gamma(G)$. We write the shortest path between U and V:

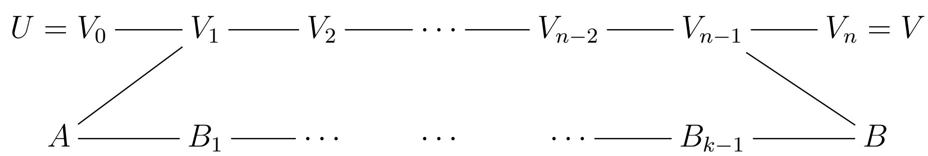
$$U = V_0 \text{ —— } V_1 \text{ —— } V_2 \text{ —— } \cdots \text{ —— } V_{n-2} \text{ —— } V_{n-1} \text{ —— } V_n = V.$$

We need to show that there exist two non-trivial proper subgroups A and B of G whose orders are prime numbers such that $d(A, B) = n$.

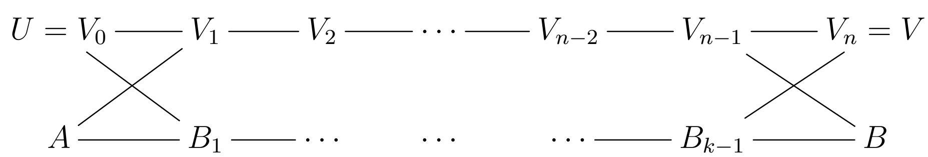
Since $U \cap V_1 > \{1\}$, we have a subgroup A of $V_0 \cap V_1$ such that A has the prime order. Similarly, we have a subgroup $B \le V_{n-1} \cap V_n$ of prime order.



It is routine to verify that $d(A, B) \le d(U, V) = n$. We claim that $d(A, B) = n$. In fact, if there is a path between A and B whose length is $k < n$ as follows:



Since A and B are prime-order groups, we have $A \le B_1$ and $B \le B_{k-1}$, which implies $U \cap B_1 \ge A$ and $V \cap B_{k-1} \ge B$. Therefore, U and B_1 are adjacent as well as V and B_{k-1}.



By replacing edges $A - B_1$ and $B - B_{k-1}$ with $V_0 - B_1$ and $V_n - B_{k-1}$, respectively, we obtain a path from U to V whose length is $k < n$

$$U = V_0 \text{ --- } B_1 \text{ --- } \cdots \text{ --- } B_{k-1} \text{ --- } V,$$

which contradicts the minimum length of the first path. To sum up, $d(A, B) = n = \delta(\Gamma(G))$. □

Remark 2.1.5. In the above proof, there is an argument that we utilize the hypothesis that $\Gamma(G)$ is not complete. By reviewing carefully, we realize that it is possible to replace edges in order to obtain the last path only if there exist these edges, or equivalently there exist intermediate vertices B_1 and B_{k-1} between A and B. This means $A \ne B_{k-1}$ and $B_1 \ne B$, or equivalently A and B are not adjacent. This does not occur because

$A \cap B \subseteq U \cap V = \{1\}$. To sum up, the last path of length k will not exist and the above proof will be wrong unless we have the hypothesis.

On the other hand, we can also verify routinely that the statement of Proposition 2.1.4 is wrong if we removing the hypothesis. In fact, if $\Gamma(G)$ is complete, then $\delta(\Gamma(G)) = 1$ and the maximum distance between two vertices of prime order is 0 (since G has only one proper subgroup of prime order).

2.2 The diameter of $\Gamma(\mathrm{GL}_n(\mathbb{F}))$ over a finite field $\mathbb{F}$

It is routine to verify that the group $\mathrm{GL}_n(\mathbb{F})$ has more than one non-trivial proper subgroup when $\mathbb{F}$ is a field containing at least three elements and $n \geq 2$. This allows us to establish its intersection graph.

In previous chapter, we knew that the diameter of a graph is defined only if this graph is connected. Therefore, it is necessary to show $\Gamma(\mathrm{GL}_n(\mathbb{F}))$ is connected. In the following theorem, we not only prove this statement, but also indicate the structure of the intersection graph. To be more specific, we can regard the intersection graph of $\mathrm{GL}_n(\mathbb{F})$ as a "circle" and the special linear graph $\mathrm{SL}_n(\mathbb{F})$ as its "center" with the radius being 2.

Theorem 2.2.1. *If $\mathbb{F}$ is a finite field containing at least three elements and $n \geq 2$, then $d(A, \mathrm{SL}_n(\mathbb{F})) \leq 2$ for every $A \in \mathcal{V}_{\mathrm{GL}_n(\mathbb{F})}$. As a direct consequence, $\Gamma(\mathrm{GL}_n(\mathbb{F}))$ is connected.*

Proof. Since $\mathbb{F}$ contains at least three elements, $\mathrm{SL}_n(\mathbb{F})$ is a non-trivial proper subgroup of $\mathrm{GL}_n(\mathbb{F})$, which means that $\mathrm{SL}_n(\mathbb{F}) \in \mathcal{V}_{\mathrm{GL}_n(\mathbb{F})}$. Let A be a non-trivial proper subgroup different from $\mathrm{SL}_n(\mathbb{F})$. We claim that $d(A, \mathrm{SL}_n(\mathbb{F})) \leq 2$. We are going to prove this statement through following steps:

Step 1: We narrow the range of subgroups that be considered. It is sufficient to focus on cases in which $A \cap \mathrm{SL}_n(\mathbb{F}) = \{\mathrm{I}_n\}$, otherwise $d(A, \mathrm{SL}_n(\mathbb{F})) = 1$. In addition, for any elements $a \in A \setminus \{\mathrm{I}_n\}$ and $b \in \mathrm{SL}_n(\mathbb{F}) \setminus \{\mathrm{I}_n\}$, if $\langle a, b\rangle < \mathrm{GL}_n(\mathbb{F})$, then we obtain the following path

$$A \text{ ---} \langle a, b\rangle \text{--- } \mathrm{SL}_n(\mathbb{F}).$$

Thus, we also have $d(A, \mathrm{SL}_n(\mathbb{F})) \leq 2$. As a result, we concentrate on possible remaining cases in which $A \cap \mathrm{SL}_n(\mathbb{F}) = \{\mathrm{I}_n\}$ and $\mathrm{GL}_n(\mathbb{F}) = \langle a, b\rangle$ for every $a \in A \setminus \{\mathrm{I}_n\}$ and $b \in \mathrm{SL}_n(\mathbb{F}) \setminus \{\mathrm{I}_n\}$.

Step 2: We research the structure of A. Via the surjective homomorphism

$$\begin{array}{rcl} \det : \mathrm{GL}_n(\mathbb{F}) & \to & \mathbb{F}^* \\ M & \mapsto & \det(M), \end{array}$$

we obtain that $\mathbb{F}^* = \langle \det(a), \det(b) \rangle = \langle \det(a) \rangle$ for every $a \in A \setminus \{\mathrm{I}_n\}$.

Let q be a prime divisor of $|\mathbb{F}^*|$. If $a^q \neq \mathrm{I}_n$, then $\mathbb{F}^* = \langle \det(a^q) \rangle = \langle \det(a)^q \rangle$. However, this case does not appear because $(q, |\mathbb{F}^*|) = q$ by (iii) of Proposition 1.2.13. Therefore, $a^q = \mathrm{I}_n$ for every $a \in A \setminus \{\mathrm{I}_n\}$.

Now, we fix an element $a \in A \setminus \{\mathrm{I}_n\}$. For every $a_1 \in A$, there is an integer l such that $\det(a_1) = \det(a)^l$ as $\mathbb{F}^* = \langle \det(a) \rangle$, which implies that $\det(a_1 a^{-l}) = 1$, or equivalently $a_1 a^{-l} \in \mathrm{SL}_n(\mathbb{F})$. The condition that $A \cap \mathrm{SL}_n(\mathbb{F}) = \{\mathrm{I}_n\}$ allows $a_1 a^{-l} = \mathrm{I}_n$, or equivalently $a_1 \in \langle a \rangle$. Thus, it must satisfy that $A = \langle a \rangle$, a group of prime order q.

Step 3: We find an intermediate vertex that can be adjacent to both A and $\mathrm{SL}_n(\mathbb{F})$. Since $\mathrm{GL}_n(\mathbb{F}) = \langle a, b \rangle = \langle a^i, b \rangle$ is non-abelian for every $1 \leq i \leq q-1$, we have a^i does not belong to $Z(\mathrm{GL}_n(\mathbb{F}))$, so $A \cap Z(\mathrm{GL}_n(\mathbb{F})) = \{\mathrm{I}_n\}$. As a consequence, by putting $B = \langle A, Z(\mathrm{GL}_n(\mathbb{F})) \rangle$, an abelian group, we obtain that $A < B < \mathrm{GL}_n(\mathbb{F})$. This means that B is a vertex of $\Gamma(\mathrm{GL}_n(\mathbb{F}))$ and adjacent to A.

Step 4: We claim that B and $\mathrm{SL}_n(\mathbb{F})$ are adjacent. It is obvious that $B \neq \mathrm{SL}_n(\mathbb{F})$ as $a \notin \mathrm{SL}_n(\mathbb{F})$. There are two following cases according to whether q is a divisor of n or not.

If q is a divisor of n, then the determinant of an element $\det(a)\mathrm{I}_n \in Z(\mathrm{GL}_n(\mathbb{F}))$ is $\det(a)^n = 1$. Thus, $\mathrm{I}_n \neq \det(a)\mathrm{I}_n \in B \cap \mathrm{SL}_n(\mathbb{F})$, so B and $\mathrm{SL}_n(\mathbb{F})$ are adjacent.

Otherwise, q is not a divisor of n, which implies that $(q, n) = 1$ because q is a prime number. As a result, there exist $u, v \in \mathbb{Z}$ such that $uq + vn = 1$. Now, we consider an element $a \cdot (\det(a)^{-v}\mathrm{I}_n) \in B$ whose determinant is $\det(a)\det(a^{-vn}) = \det(a^{1-vn}) = \det(a^{uq}) = 1$, so $a\det(a)^{-v}\mathrm{I}_n \in \mathrm{SL}_n(\mathbb{F})$. The fact that $a \notin Z(\mathrm{GL}_n(\mathbb{F}))$ implies $a(\det(a)^{-v}\mathrm{I}_n) \neq \mathrm{I}_n$. Therefore, we obtain $B \cap \mathrm{SL}_n(\mathbb{F}) \neq \{\mathrm{I}_n\}$.

After considering both cases, we usually obtain the path

$$A \text{ --- } B \text{ --- } \mathrm{SL}_n(\mathbb{F}).$$

To sum up, $d(A, \mathrm{SL}_n(\mathbb{F})) \leq 2$ for every $A \in \mathcal{V}_{\mathrm{GL}_n(\mathbb{F})}$ and $A \neq \mathrm{SL}_n(\mathbb{F})$. The consequence that $\Gamma(\mathrm{GL}_n(\mathbb{F}))$ is connected can be implied straightforwardly. $\square$

Theorem 2.2.1 allows us to show the diameter of $\Gamma(\mathrm{GL}_n(\mathbb{F}))$ is at most 4 straightforwardly. Meanwhile, we can prove a greater inequality, with the supremum of $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F})))$ being 3, in the following theorem.

Theorem 2.2.2. *If $\mathbb{F}$ is a finite field containing at least three elements and $n > 1$, then* $2 \leq \delta(\Gamma(\mathrm{GL}_n(\mathbb{F}))) \leq 3$.

Proof. Since $\mathbb{F}$ is finite, $\mathrm{GL}_n(\mathbb{F})$ is finite. We are going to show the theorem through steps.

Step 1: We claim that $\mathrm{GL}_n(\mathbb{F})$ is not complete. In fact, put $c = \mathrm{I}_n + e_{12}$ and $d = \mathrm{I}_n + e_{21}$ where e_{ij} is the matrix whose (i, j)-th entry is 1 and all other entries are zeros. For $p = \mathrm{Char}(\mathbb{F})$, it is routine to verify that $\langle c\rangle = \{\mathrm{I}_n + me_{12} \mid 0 \leq m \leq p - 1\}$ and $\langle d\rangle = \{\mathrm{I}_n + me_{21} \mid 0 \leq m \leq p - 1\}$. This implies that $\langle c\rangle \cap \langle d\rangle = \{\mathrm{I}_n\}$. Thus, $\langle c\rangle$ and $\langle d\rangle$ are not adjacent, or equivalently $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F}))) \geq 2$.

Step 2: We claim that $d(A, B) \leq 3$ for every two distinct non-trivial proper subgroups A, B of prime order of $\mathrm{GL}_n(\mathbb{F})$. Put $A_1 = \langle A, Z(\mathrm{GL}_n(\mathbb{F}))\rangle > \{\mathrm{I}_n\}$ and $B_1 = \langle B, Z(\mathrm{GL}_n(\mathbb{F}))\rangle > \{\mathrm{I}_n\}$. It is obvious that A_1 and B_1 are abelian, which implies both A_1 and B_1 differ from $\mathrm{GL}_n(\mathbb{F})$ because $\mathrm{GL}_n(\mathbb{F})$ is non-abelian. We have just shown that $A_1, B_1 \in \mathcal{V}_{\mathrm{GL}_n(\mathbb{F})}$. It is routine to verify that we have the (not necessarily simple) path

$$A \text{ —— } A_1 \text{ —— } B_1 \text{ —— } B.$$

As a result, we obtain $d(A, B) \leq 3$ for every subgroups A and B of prime order.

Step 3: We find possible values of the diameter of the graph. By the conclusion of previous steps and Proposition 2.1.4, the diameter of $\Gamma(\mathrm{GL}_n(\mathbb{F}))$ is at most 3. To sum up, we have $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F}))) \in \{2, 3\}$. □

A question raise naturally is if we could make the inequality in Theorem 2.2.2 stricter since if we can, we know the precise value of the diameter. The answer is it's not possible to find a stricter one and we are going to show this statement by providing some examples in the later section.

2.3 A sufficient condition determining the diameter of $\Gamma(\mathrm{GL}_n(\mathbb{F}))$

Theorem 2.2.2 enables us to know that the diameter of a general linear group is either 2 or 3. Meanwhile, it has been impossible to verify what $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F})))$ is exactly for each number n and each field $\mathbb{F}$. In the dissertation, we only can establish general criteria for measuring the diameter of the intersection graphs of $\mathrm{GL}_n(\mathbb{F})$.

Theorem 2.3.1. *Let $\mathbb{F}$ be a finite field containing at least three elements and let $n \geq 2$. Then, $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F}))) = 3$ if and only if $\mathrm{GL}_n(\mathbb{F})$ is generated by two elements of prime order.*

Proof. Assume that $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F}))) = 3$. By Proposition 2.1.4, there exist two non-trivial proper subgroups A, B of prime order such that $d(A,B) = 3$. If $\langle A, B\rangle < \mathrm{GL}_n(\mathbb{F})$, then we have the following path

$$A \longrightarrow \langle A, B\rangle \longrightarrow B,$$

so $d(A,B) \leq 2$, a contradiction! As a result, $\langle A, B\rangle = \mathrm{GL}_n(\mathbb{F})$.

Conversely, assume that $\mathrm{GL}_n(\mathbb{F}) = \langle A, B\rangle$, where A, B are non-trivial proper subgroups of prime order.

If $d(A,B) = 1$, which means that $A \cap B \neq \{\mathrm{I}_n\}$, then $A = B$ because of their prime orders. Thus, $\mathrm{GL}_n(\mathbb{F}) = \langle A\rangle$ is a cyclic group, a contradiction!

If $d(A,B) = 2$, then there exists a non-trivial proper subgroup G of $\mathrm{GL}_n(\mathbb{F})$ such that G is adjacent to both A and B. This means that $G \cap A > \{\mathrm{I}_n\}$ and $G \cap B > \{\mathrm{I}_n\}$. Since A and B are groups of prime order, we obtain that $A \leq G$ and $B \leq G$, so $\langle A, B\rangle \leq G$. Therefore, it must satisfy $\mathrm{GL}_n(\mathbb{F}) \leq G$, a contradiction!

Therefore, $d(A,B) \geq 3$, and by Theorem 2.2.2, we obtain $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F}))) = 3$. □

We have a direct corollary of Theorem 2.3.1 that is helpful for us to verify many cases in the next section.

Corollary 2.3.2. *Let F be a finite field containing at least three elements and let $n \geq 2$. If the multiplicative group $\mathbb{F}^* \neq \langle \alpha, \beta\rangle$ for every elements $\alpha, \beta \in \mathbb{F}^*$ of prime order, then $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F}))) = 2$.*

Proof. Because the determinant of a matrix of prime order is either one or an element of prime order, the hypothesis implies $\mathrm{GL}_n(\mathbb{F})$ is not generated by two elements of prime order via the homomorphism det. By Theorem 2.3.1, we obtain that $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F}))) \neq 3$, or equivalently $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F}))) = 2$ by Theorem 2.2.2. The proof is completed. □

2.4 Exemplifying some types of $\mathrm{GL}_n(\mathbb{F})$ according to $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F})))$

Now, we utilize Theorem 2.3.1 and Corollary 2.3.2 to measure the diameters of the intersection graphs of some general linear groups. For this purpose, we verify some groups that are generated by exactly two elements of prime order. The first one is $\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$.

Lemma 2.4.1. *The group* $\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$ *is generated by two elements* $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$.

Proof. Put $a = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, and $x(z) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}, y(z) = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}$, where $z \in \mathbb{Z}/3\mathbb{Z} = \{0, \pm 1\}$. We obtain an inclusion that $\langle x(z), y(z) \mid z \in \mathbb{Z}/3\mathbb{Z} \rangle \leq \langle a, b \rangle$ because of the equalities

$$x(1) = b^2ab^2ab; x(-1) = b^2abab; y(1) = babab^2; y(-1) = bab^2ab^2; x(0) = y(0) = \mathrm{I}_n.$$

Afterwards, we claim that $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ is a subgroup of $\langle x(z), y(z) \mid z \in \mathbb{Z}/3\mathbb{Z} \rangle$. In fact, we have the form of each element of $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ is one of two cases:

$$\begin{bmatrix} r & (rh-1)s \\ s^{-1} & h \end{bmatrix}, \begin{bmatrix} s & t \\ 0 & s^{-1} \end{bmatrix},$$

where $r, h, t \in \{0, \pm 1\}$ and $s \in \{\pm 1\}$. By performing many calculations, we obtain

$$\begin{bmatrix} r & (rh-1)s \\ s^{-1} & h \end{bmatrix} = x(rs) \cdot x(-1) \cdot y(1) \cdot x(-1) \cdot x(-s) \cdot y(s) \cdot x(-s) \cdot x(1) \cdot y(-1) \cdot x(1) \cdot x(hs),$$

$$\begin{bmatrix} s & t \\ 0 & s^{-1} \end{bmatrix} = x(st) \cdot x(-1) \cdot y(1) \cdot x(-1) \cdot x(s) \cdot y(-s) \cdot x(s).$$

As a consequence,

$$\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z}) \leq \langle x(z), y(z) \mid z \in \mathbb{Z}/3\mathbb{Z} \rangle.$$

As a result, we have the inclusion $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z}) \leq \langle a, b\rangle$. In addition, since $a \notin \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$, it must satisfy that $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z}) < \langle a, b\rangle$.

Furthermore, the fact that $|\frac{\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})}{\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})}| = |(\mathbb{Z}/3\mathbb{Z})^*| = 2$ results in $\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}) = \langle a, b\rangle$. The proof is completed. □

Proposition 2.4.2. $\delta(\Gamma(\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}))) = 3$.

Proof. It is routine to verify that $a^2 = b^3 = \mathrm{I}_2$ where $a = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$. Combining Lemma 2.4.1 with Theorem 2.3.1, we obtain that $\delta(\Gamma(\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}))) = 3$. □

Another general linear group generated by only two elements of prime order is $\mathrm{GL}_2(\mathbb{F})$ such that the order of $\mathbb{F}^*$ is a Mersenne prime, the prime number is of the form $2^t - 1$, $t \in \mathbb{N}$. People have discovered 51 Mersenne primes through over 2000 years. This means that we have had 51 particular instances of the general form.

Lemma 2.4.3. *Let $\mathbb{F}$ be a finite field of characteristic 2 containing at least 3 elements such that the order of the multiplicative group $\mathbb{F}^*$ is a prime number. If λ is a primitive element of $\mathbb{F}$, then $\mathrm{GL}_2(\mathbb{F})$ is generated by $\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. As a result, $\delta(\Gamma(\mathrm{GL}_2(\mathbb{F}))) = 3$.*

Proof. Put $a = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. By [Wat89], there exist $\alpha, \beta, \gamma \in \mathbb{F}^*$ such that $\mathrm{GL}_2(\mathbb{F})$ is generated by $\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & \beta \\ 1 & \gamma \end{bmatrix}$. Thus, it is sufficient to show that $\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & \beta \\ 1 & \gamma \end{bmatrix}$ belong to $\langle a, b\rangle$ for every $\alpha, \beta, \gamma \in \mathbb{F}^*$. In fact, since $\mathbb{F}^* = \langle \lambda \rangle$ and $a^t = \begin{bmatrix} \lambda^t & 0 \\ 0 & 1 \end{bmatrix}$ for every $0 \leq t \leq |\mathbb{F}^*| - 1$, we have $\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \in \langle a \rangle \leq \langle a, b\rangle$ for every $\alpha \in \mathbb{F}^*$. On the other hand, it is routine to check that

$$a^{-4}bab^2a^2ba^2b^2a^2bab^2 = \begin{bmatrix} 1 & 0 \\ 0 & \lambda^4 \end{bmatrix}.$$

Since the order of λ is $|\mathbb{F}^*|$, an odd number, and $(4, |\mathbb{F}^*|) = 1$, the element λ^4 is also a primitive element of $\mathbb{F}^*$. Thus,

$$\begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} \in \left\langle \begin{bmatrix} 1 & 0 \\ 0 & \lambda^4 \end{bmatrix} \right\rangle \leq \langle a, b\rangle$$

for every $\gamma \in \mathbb{F}^*$. Therefore, for every $\beta, \gamma \in \mathbb{F}^*$,

$$\begin{bmatrix} 0 & \beta \\ 1 & \gamma \end{bmatrix} = \begin{bmatrix} \beta\gamma^{-1} & 0 \\ 0 & 1 \end{bmatrix} b^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} \in \langle a, b\rangle.$$

We have just shown that $\mathrm{GL}_2(\mathbb{F}) = \langle a, b\rangle$. It is routine to check that $a^{|\mathbb{F}^*|} = b^3 = \mathrm{I}_2$, so $\delta(\Gamma(\mathrm{GL}_2(\mathbb{F}))) = 3$ by Theorem 2.3.1. The proof is completed. □

Next, we mention the definition of the prime omega function, the function counting the total number of prime factors (with multiplicity) of a natural numbers greater than 1 with the aim of expressing Theorem 2.4.5.

Definition 2.4.4. The *prime Omega function*, denoted by Ω, is defined as follows: for every integer n greater than 1 with its prime decomposition is $\prod_{i=1}^{t} p_i^{k_i}$, the prime Omega function of n is $\Omega(n) = \sum_{i=1}^{t} k_i$.

Now, we reach the main theorem of the section which indicates some cases of $\mathbb{F}$ and n in which $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F})))$ is determined exactly.

Theorem 2.4.5. *Let $\mathbb{F}$ be a finite field containing at least three elements and $n \geq 2$. Then we have the following conclusions of the diameter of the intersection graph of $\mathrm{GL}_n(\mathbb{F})$:*

(i) If $\Omega(|\mathbb{F}^*|) \geq 3$, then $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F}))) = 2$;

(ii) If $\Omega(|\mathbb{F}^*|) = 2$ and $|\mathbb{F}^*| = h^2$ for some prime number h, then $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F}))) = 2$;

(iii) If $\Omega(|\mathbb{F}^*|) = 1$, then $\delta(\Gamma(\mathrm{GL}_2(\mathbb{F}))) = 3$.

Proof. (i) For arbitrary α and β are elements of $\mathbb{F}^*$ with orders being prime numbers h and k, respectively, the order of $\langle \alpha, \beta\rangle$ is either h (when $\alpha \in \langle \beta\rangle$) or hk (when $\alpha \notin \langle \beta\rangle$). In both cases, we have $\Omega(|\langle \alpha, \beta\rangle|) \leq 2 < \Omega(|\mathbb{F}^*|)$, which results in $\mathbb{F}^* \neq \langle \alpha, \beta\rangle$. By Corollary 2.3.2, we obtain $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F}))) = 2$.

(ii) Similarly, we aim to show that $\Omega(|\langle \alpha, \beta\rangle|) = 1 < \Omega(|\mathbb{F}^*|)$. In fact, if α and β are elements of prime order, then their orders are h each by (ii) of Proposition 1.2.13. As a result, we have $\langle \alpha\rangle = \langle \beta\rangle$ as $\mathbb{F}^*$ is cyclic (by (iv) of Proposition 1.2.13). Therefore, we have $\Omega(\langle \alpha, \beta\rangle) = \Omega(\langle \alpha\rangle) = \Omega(h) = 1$.

(iii) Let $p = \mathrm{Char}(\mathbb{F})$ and write $|\mathbb{F}| = p^n$ for some $n \in \mathbb{N}$. The condition $\Omega(|\mathbb{F}^*|) = 1$ is equivalent to $p^n - 1$ is a prime number. If $p > 2$, then $p^n - 1$ is even, which implies that $p^n - 1 = 2$, or equivalently $p = 3$ and $n = 1$. In this case, $\mathbb{F} \simeq \mathbb{Z}/3\mathbb{Z}$. By

Proposition 2.4.2, we have $\delta(\Gamma(\mathrm{GL}_2(\mathbb{F}))) = 3$. Otherwise, when $p = 2$, by Lemma 2.4.3, we also reach the same conclusion.

□

There are many fields $\mathbb{F}$ such that $\Omega(|\mathbb{F}^*|) \geq 3$. However, $\mathbb{Z}/5\mathbb{Z}$ (up to an isomorphism) is an unique group satisfying (ii) of Theorem 2.4.5. As for the last conclusion, there have been 52 groups satisfying (iii), including $\mathbb{Z}/3\mathbb{Z}$ and 51 groups related to Mersenne primes.

Theorem 2.4.5 shows the inequality in Theorem 2.2.2 is as strict as possible.

2.5 End-equivalence of $\Gamma(\mathrm{GL}_n(\mathbb{F}))$ when $\mathbb{F}$ is infinite

For infinite field $\mathbb{F}$, the diameter of $\Gamma(\mathrm{GL}_n(\mathbb{F}))$ is exactly 2 (Proposition 2.5.2). In the order words, the problem related to diameter of the intersection graphs is handled totally.

Lemma 2.5.1. *Every general linear group of degree greater than* 1 *over an infinite field is infinitely generated. As a direct result, the intersection graph of the group is infinite.*

Proof. This is a particular case of [NBH17, Theorem 5.4]. □

Proposition 2.5.2. The intersection graph of any general linear group of degree n greater than 1 over an infinite field $\mathbb{F}$ is connected with the diameter being 2.

Proof. Let A and B be two distinct non-trivial proper subgroups of $\mathrm{GL}_n(\mathbb{F})$. If either $A \subseteq B$ or $B \subseteq A$, then $A \cap B \neq \{\mathrm{I}_n\}$, which allows A and B to be adjacent. Otherwise, since $A \nsubseteq B$ and $B \nsubseteq A$, there exist $a \in A \setminus B$ and $b \in B \setminus A$. By Lemma 2.5.1, we have $\langle a, b\rangle \neq \mathrm{GL}_n(\mathbb{F})$, which means that $\langle a, b\rangle$ is a vertex of $\Gamma(\mathrm{GL}_n(\mathbb{F}))$. The fact that $\{\mathrm{I}_n\} < \langle a\rangle \leq A \cap \langle a, b\rangle$ and $\{\mathrm{I}_n\} < \langle b\rangle \leq B \cap \langle a, b\rangle$ implies straightforwardly

$$A \text{ --- } \langle a, b\rangle \text{ --- } B.$$

Therefore, $d(A, B) \leq 2$ for every distinct non-trivial proper subgroups A, B of $\mathrm{GL}_n(\mathbb{F})$, which means that $\Gamma(\mathrm{GL}_n(\mathbb{F}))$ is connected and its diameter is at most 2.

We need to prove that $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F}))) = 2$ by indicating two vertices whose distance is 2. In fact, we consider two non-trivial proper subgroups $U = \{\mathrm{I}_n + me_{12} \mid m \in \mathbb{F}\}$ and $V = \{\mathrm{I}_n + me_{21} \mid m \in \mathbb{F}\}$. It is routine to verify $U \cap V = \{\mathrm{I}_n\}$, or equivalently $d(U, V) \geq 2$. To sum up, $\delta(\Gamma(\mathrm{GL}_n(\mathbb{F}))) = 2$. □

In order to broadening perspective on cases of infinite groups, we investigate a characteristic of infinite graphs, this is the end-equivalence relation of rays of $\Gamma(\mathrm{GL}_n(\mathbb{F}))$ (Theorem 2.5.4). As well as the important role of groups of prime order in previous sections, the infinitely generated groups is the kernel of the proofs of two following results.

Lemma 2.5.3. *If G is a non-Noetherian group, then $\Gamma(G)$ contains infinitely many disjoint rays.*

Proof. By Proposition 1.2.15, there exists an infinite chain of non-trivial subgroups of G as follows

$$G_1 < G_2 < G_3 < \cdots < G_n < \cdots .$$

Denote by p_i the i-th prime number. We obtain a ray

$$\Gamma_i : G_{p_i} \text{ —— } G_{p_i^2} \text{ —— } G_{p_i^3} \text{ —— } \cdots \text{ —— } G_{p_i^n} \text{ —— } \cdots,$$

for each i. Since there is no power of p_i being a power of p_j when $i \neq j$, Γ_i and Γ_j are disjoint. It is well-known that there are infinitely many prime numbers, which implies there are infinitely many rays Γ_i's. □

Now, we reach the main theorem ending the dissertation.

Theorem 2.5.4. *If $\mathbb{F}$ is an infinite field and $n \geq 2$, then $\Gamma(\mathrm{GL}_n(\mathbb{F}))$ is one-ended and the end is thick.*

Proof. By Lemma 2.5.1, $\mathrm{GL}_n(\mathbb{F})$ is infinitely generated. Therefore, by Proposition 1.2.15, we obtain that $\mathrm{GL}_n(\mathbb{F})$ is non-Noetherian, so $\Gamma(\mathrm{GL}_n(\mathbb{F}))$ has infinitely many disjoint rays by Lemma 2.5.3. According to Definition 1.1.13, it is sufficient to show that all rays of $\Gamma(\mathrm{GL}_n(\mathbb{F}))$ are end-equivalent. Let $(A_i)_{i\geq 0}$ and $(B_i)_{i\geq 0}$ be two arbitrary rays of $\Gamma(\mathrm{GL}_n(\mathbb{F}))$.

$$A_0 \text{ —— } A_1 \text{ —— } A_2 \text{ —— } \cdots \text{ —— } A_n \text{ —— } \cdots$$

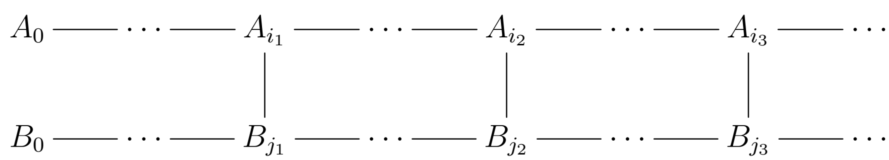
$$B_0 \text{ —— } B_1 \text{ —— } B_2 \text{ —— } \cdots \text{ —— } B_n \text{ —— } \cdots$$

We are going to show $(A_i)_{i\geq 0} \equiv (B_i)_{i\geq 0}$ by finding a ray $(C_i)_{i\geq 0}$ of $\Gamma(\mathrm{GL}_n(\mathbb{F}))$ that has infinitely many vertices in common with $(A_i)_{i\geq 0}$ as well as $(B_i)_{i\geq 0}$ (Proposition 1.1.14). All general cases of $(A_i)_{i\geq 0}$ and $(B_i)_{i\geq 0}$ can be considered as three particular cases:

Case 1: If $\{A_i \mid i \geq 0\} \cap \{B_i \mid i \geq 0\}$ is infinite, then $(A_i)_{i\geq 0} \equiv (B_i)_{i\geq 0}$ since we can choose $C_i = A_i$ for all $i \geq 0$.

Otherwise, we concentrate on cases in which $\{A_i \mid i \geq 0\} \cap \{B_i \mid i \geq 0\}$ is finite, which means that there exists $k \in \mathbb{N}$ such that $\{A_i \mid i \geq k\} \cap \{B_i \mid i \geq k\} = \emptyset$. The fact that $(A_i)_{i\geq 0} \equiv (A_i)_{i\geq k}$ (Remark 1.1.15) and the transitivity of end-equivalence (in Proposition 1.1.11) enables us to assume that $\{A_i \mid i \geq 0\} \cap \{B_i \mid i \geq 0\} = \emptyset$ without loss of generality. In the other words, two rays $(A_i)_{i\geq 0}$ and $(B_i)_{i\geq 0}$ are disjoint. With this assumption, we have two following cases.

Case 2: For each $k \geq 1$, there exist $i, j \geq k$ such that A_i is adjacent to B_j. As a result, we can choose two increasing infinite sequences of natural numbers $(i_z)_{z\geq 1}$ and $(j_z)_{z\geq 1}$ such that A_{i_z} and B_{j_z} are adjacent for every $z \geq 1$.



We choose the path $(C_i)_{i\geq 0}$ which is

$$(A_{i_1} \text{——} B_{j_1}) \circ (B_z)_{j_1 \leq z \leq j_2} \circ (B_{j_2} \text{——} A_{i_2}) \circ (A_z)_{i_2 \leq z \leq i_3} \circ (A_{i_3} \text{——} B_{j_3}) \circ \cdots.$$

Since $\{A_i \mid i \geq 0\} \cap \{B_i \mid i \geq 0\} = \emptyset$, all elements of $(C_i)_{i\geq 0}$ are distinct, or equivalently $(C_i)_{i\geq 0}$ is a ray. It is routine to verify that $\{C_i \mid i \geq 0\} \cap \{A_i \mid i \geq 0\}$ is infinite as well as $\{C_i \mid i \geq 0\} \cap \{B_i \mid i \geq 0\}$. By Proposition 1.1.14, we have $(A_i)_{i\geq 0} \equiv (B_i)_{i\geq 0}$.

Case 3: If there exists $k \geq 0$ such that A_i and B_j are not adjacent for every $i, j \geq k$. Repetition of the above argument allows us to assume that A_i and B_j are not adjacent for every $i, j \geq 0$, without loss of generality. With this additional assumption, we are going to construct an intermediate ray $(D_i)_{i\geq 0}$ by induction.

Because A_i and B_i are not adjacent, we have $A_i \not\subseteq B_i$ and $B_i \not\subseteq A_i$, which enables us to choose $a_i \in A_i \setminus B_i$ and $b_i \in B_i \setminus A_i$ for every $i \geq 0$. Put $d_0 = \mathrm{I}_n$ and $D_0 = \langle a_0, b_0 \rangle$. For each $i \geq 0$, we assume that the subgroup D_i was finitely generated. As a result, we obtain $D_i < \mathrm{GL}_n(\mathbb{F})$ (by Lemma 2.5.1), which enables us to choose an element $d_{i+1} \in \mathrm{GL}_n(\mathbb{F}) \setminus D_i$. Then, we put

$$D_{i+1} = \langle D_i, a_{i+1}, b_{i+1}, d_{i+1} \rangle.$$

As a result, D_{i+1} is also finitely generated, so we can construct the chain $(D_i)_{i\geq 0}$ inductively.

Now, we list some properties of the chain $(D_i)_{i\geq 0}$.

(i) We obtain that $D_0 < D_1 < D_2 < \cdots$, because $d_{i+1} \notin D_i$. In particular, $D_i \neq D_j$ for every $i \neq j$. As a result, the path

$$D_0 \text{ ------ } D_1 \text{ ------ } D_2 \text{ ------ } \cdots$$

is a ray.

(ii) For every $i \geq 0$, D_i is adjacent to both A_i and B_i because $\langle a_i \rangle \leq D_i \cap A_i$ and $\langle b_i \rangle \leq D_i \cap B_i$.

$$\begin{array}{ccccccccccc}
A_0 & \text{------} & A_1 & \text{------} & A_2 & \text{------} & \cdots & \text{------} & A_n & \text{------} & \cdots \\
| & & | & & | & & & & | & & \\
D_0 & \text{------} & D_1 & \text{------} & D_2 & \text{------} & \cdots & \text{------} & D_n & \text{------} & \cdots \\
| & & | & & | & & & & | & & \\
B_0 & \text{------} & B_1 & \text{------} & B_2 & \text{------} & \cdots & \text{------} & B_n & \text{------} & \cdots
\end{array}$$

(iii) Three rays $(D_i)_{i\geq 0}$, $(A_i)_{i\geq 0}$ and $(B_i)_{i\geq 0}$ are disjoint. In fact, if there exists z such that $D_z = A_j$ for some $j \geq 0$, which implies $A_j \cap B_z = D_z \cap B_z \neq \{\mathrm{I}_n\}$. Thus, we have either $A_j = B_z$ or A_j and B_z are adjacent, which contradicts two assumptions of this case. As a result, $(D_i)_{i\geq 0}$ and $(A_i)_{i\geq 0}$ are disjoint. For $(D_i)_{i\geq 0}$ and $(B_i)_{i\geq 0}$, we show similarly.

Finally, we choose a path $(C_i)_{i\geq 0}$ as follows

$$(C_i)_{i\geq 0} = A_0 \text{ ------ } D_0 \text{ ------ } B_0 \text{ ------ } B_1 \text{ ------ } D_1 \text{ ------ } A_1 \text{ ------ } A_2 \text{ ------ } D_2 \text{ ------ } B_2 \text{ ------ } \cdots.$$

By the properties (i) and (iii), all elements of $(C_i)_{i\geq 0}$ are distinct, or equivalently $(C_i)_{i\geq 0}$ is a ray. In addition, we have that both sets $\{C_i \mid i \geq 0\} \cap \{A_i \mid i \geq 0\}$ and $\{C_i \mid i \geq 0\} \cap \{B_i \mid i \geq 0\}$ are infinite. Therefore, $(A_i)_{i\geq 0} \equiv (B_i)_{i\geq 0}$ by Proposition 1.1.14. The proof is completed. □

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